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Integrals of the Motion, Green Functions, and Coherent States of Dynamical Systems

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Abstract

The connection between the integrals of the motion of a quantum system and its Green function is established. The Green function is shown to be the eigenfunction of the integrals of the motion which describe initial points of the system trajectory in the phase space of average coordinates and moments. The explicit expressions for the Green functions of the N-dimensional system with the Hamiltonian which is the most general quadratic form of coordinates and momenta with time-dependent coefficients is obtained in coordinate, momentum, and coherent states representations. The Green functions of the nonstationary singular oscillator and of the stationary Schrödinger equation are also obtained.

1. *Introduction*

In the present article we want to elucidate on the connection between the integrals of the motion of a quantum system and its Green function. This connection proves to be very simple, almost trivial: the Green function is the eigenfunction of the integrals of the motion describing initial points of the system trajectory in the phase space. Nevertheless we want to discuss this problem in detail for two reasons. The first reason is that this problem, in spite of its simplicity, was not considered distinctly in any of available textbooks or original papers. (Some notes on this problem were made by Malkin and Man'ko (1970; 1971) and Aronson *et al.* (1974); in the last paper the connection of the integrals of the motion with the concept of the dynamical symmetry was also discussed.) The second reason is that using the equations connecting the Green function and integrals of the motion one can obtain the explicit expressions for the Green functions of some systems which are undoubtedly very interesting from the viewpoint of physics.

Particularly, using this approach we shall obtain the Green function in the case when the Hamiltonian is the most general (i.e. including linear terms) quadratic form of coordinates and momenta with time-dependent coefficients.

Earlier, several authors considered only special examples of general quadratic systems. For example, general quadratic systems with time-dependent coefficients but without linear terms were considered by Chernikov (1967) and Popov (1973). The Green function of the most general quadratic system was obtained for the first time by Malkin *et al.* (1971; 1973) with the aid of the coherent states method, but it was not given in the form convenient for applications. Coherent states for this problem were also constructed by Holz (1970).

The problem of calculating the Green function of a quadratic system has much in common with the problem of calculating the kernel of the unitary transformation of the Hilbert space of quantum states corresponding to some linear canonical transformation of operators acting in this space. Berezin (1966) has investigated in detail the last problem in the case of finite- and infinite-dimensional systems of operators satisfying both Bose's and Fermi's commutation relations. Later the analogous problem (in the case of the finite system of coordinate and momentum-type operators) was considered by Moshinsky and Quesne (1971), Boon and Seligman (1973) and Wolf (1974).

Our results are formally analogous to that obtained in these papers. However, there is a significant physical difference. In all mentioned papers the linear transformations of the operators were considered as given *a priori*, while in the present paper these transformations are to be determined by the Hamiltonian of the system. The second difference is that in the papers mentioned above the kernels of the unitary transformations were calculated up to an arbitrary phase factor, while the Green functions obtained in this paper are determined uniquely.

Besides the general quadratic system we shall also consider the singular nonstationary oscillator to illustrate once more general relations. After this we shall calculate the Green functions of the stationary Schrödinger equation for some simple systems.

2. *Integrals of the Motion and the Green Function*

Let us consider an arbitrary quantum system with N degrees of freedom. The time evolution of the wave function $\Psi(\mathbf{x}, t)$ describing any pure state of this system is completely determined by the Green function $G(\mathbf{x}, \mathbf{y}; t)$ according to the relation

$$\Psi(\mathbf{x}, t) = \int G(\mathbf{x}, \mathbf{y}; t)\Psi(\mathbf{y}, 0)\, dy = \hat{U}\Psi(\mathbf{x}, 0)$$
$$\mathbf{x} \equiv (x_1, x_2, \ldots x_N) \tag{2.1}$$

The Green function is the kernel of the evolution operator $\hat{U}$ which satisfies the equation

$$i\hbar \frac{\partial \hat{U}}{\partial t} = \hat{H}\hat{U}; \qquad \hat{U}(0) = \hat{E} \tag{2.2}$$

where $\hat{E}$ is the unity operator, and $\hat{H}$ is the Hamiltonian of the system. The operator $\hat{H}$ may be both stationary and nonstationary; moreover, it may be nonhermitian. The only significant restriction on $\hat{H}$ is that the evolution

operator must exist and have the inverse one. If a complete system of solutions to the Schrödinger equation $\{\Psi_n(\mathrm{x}, t)\}$ is known, then the Green function can be expressed as

$$G(\mathrm{x}, \mathrm{y}; t) = \sum_n \Psi_n(\mathrm{x}, t)\Psi_n^*(\mathrm{y}, 0)$$

However this method of calculating the Green function is not the most convenient one, because it requires the knowledge of all functions Ψ_n. We want to show that the Green function can be obtained with the aid of a much more convenient procedure, provided the integrals of the motion are known.

We define the integral of the motion as such operator $\hat{I}$ that transforms every solution of the Schrodinger equation into a solution of the same equation. This means that $\hat{I}$ must satisfy in the space of the solutions of the Schrödinger equation the equation

$$\left[i\hbar \frac{\partial}{\partial t} - \hat{H}, \hat{I} \right] \Psi = 0 \tag{2.3}$$

If the Hamiltonian is a hermitian operator, an integral of the motion has the property that its average values (averaged with respect to solutions of the Schrödinger equation) do not change in time: $\partial/\partial t \int \Psi^* \hat{I} \Psi \, d\mathrm{x} = 0$. For non-hermitian Hamiltonians this property does not hold: an operator whose average values do not change in time must satisfy not equation (2.3), but the following one: $i\hbar(\partial/\partial t)I = H^+\hat{I} - \hat{I}\hat{H}$ (the cross means the hermitian conjugation).

Now let us note that every quantum system with N degrees of freedom has $2N$ independent integrals of the motion. This statement follows immediately the correspondence principle, since every classical system has $2N$ independent integrals of the motion. One can choose, e.g., the initial values of coordinates and momenta as such independent integrals of the motion. The corresponding quantum integrals of the motion can be expressed as follows:

$$\hat{\mathbf{X}} = \hat{U}\hat{\mathbf{x}}\hat{U}^{-1}; \qquad \hat{\mathbf{P}} = \hat{U}\hat{\mathbf{p}}\hat{U}^{-1} \tag{2.4}$$

(one can easily verify that the operators (2.4) satisfy equation (2.3) due to equation (2.2)). It is convenient to introduce the $2N$-dimensional vectors $\mathbf{q} = (\mathbf{p}, \mathbf{x})$ and $\mathbf{Q} = (\mathbf{P}, \mathbf{X})$. Then two equations (2.4) can be combined into one relation $\hat{\mathbf{Q}} = \hat{U}\hat{\mathbf{q}}\hat{U}^{-1}$. The commutation relations between operators $\hat{Q}_j$ are the same as between $\hat{\mathbf{q}}_j$, although the operators $\hat{Q}_j$ can be nonhermitian (if the Hamiltonian is nonhermitian).

Let us note now that at the initial moment the Green function equals the delta-function:

$$G(\mathrm{x}, \mathrm{y}; 0) = \delta(\mathrm{x} - \mathrm{y}); \qquad G(\mathrm{x}, \mathrm{y}; t) = \hat{U}_{\{1\}}\delta(\mathrm{x} - \mathrm{y}) \tag{2.5}$$

and that the action of the operator $\hat{\mathbf{q}}$ on $\delta(\mathrm{x} - \mathrm{y})$ as the function of the first argument x is equivalent to the action of the transposed operator $\hat{\mathbf{q}}^T$ on $\delta(\mathrm{x} - \mathrm{y})$ as the function of y

$$\hat{\mathbf{q}}_{\{1\}}\delta(\mathrm{x} - \mathrm{y}) = \hat{\mathbf{q}}^T_{\{2\}}\delta(\mathrm{x} - \mathrm{y}) \tag{2.6}$$

The symbol $\hat{I}_{\{1,2\}}\psi(\mathbf{x}, \mathbf{y})$ means that the operator $\hat{I}$ acts on ψ as the function of the first or second variable, respectively, while another variable should be considered as a parameter. In the coordinate representation $\hat{\mathbf{x}}^T = \hat{\mathbf{x}}$, $\hat{\mathbf{p}}^T = -\hat{\mathbf{p}}$. Acting on both parts of equation (2.6) by the evolution operator $\hat{U}$ one obtains

$$\hat{U}_{\{1\}}\hat{\mathbf{q}}_{\{1\}}\delta(\mathbf{x}-\mathbf{y}) = (\hat{U}\hat{\mathbf{q}}\hat{U}^{-1})_{\{1\}}\hat{U}_{\{1\}}\delta(\mathbf{x}-\mathbf{y}) = \hat{Q}_{\{1\}}G(\mathbf{x}, \mathbf{y}; t);$$

$$\hat{U}_{\{1\}}\hat{\mathbf{q}}^T_{\{2\}}\delta(\mathbf{x}-\mathbf{y}) = \hat{\mathbf{q}}^T_{\{2\}}\hat{U}_{\{1\}}\delta(\mathbf{x}-\mathbf{y}) = \hat{\mathbf{q}}^T_{\{2\}}G(\mathbf{x}, \mathbf{y}; t)$$

Consequently, the Green function satisfies the following vector equation (really this is the system of $2N$ scalar equations):

$$\hat{\mathbf{Q}}_{\{1\}}G(\mathbf{x}, \mathbf{y}; t) = \hat{\mathbf{q}}^T_{\{2\}}G(\mathbf{x}, \mathbf{y}; t) \tag{2.7}$$

or, in a more detailed form

$$\hat{\mathbf{X}}_{\{1\}}G(\mathbf{x}, \mathbf{y}; t) = \mathbf{y}G(\mathbf{x}, \mathbf{y}; t) \tag{2.7a}$$

$$\hat{\mathbf{P}}_{\{1\}}\hat{G}(\mathbf{x}, \mathbf{y}; t) = i\hbar\frac{\partial}{\partial \mathbf{y}}G(\mathbf{x}, \mathbf{y}; t) \tag{2.7b}$$

So the Green function in the coordinate representation is the eigenfunction of the integral of the motion $\hat{\mathbf{X}}$. This fact is evidently the consequence of the physical significance of the Green function as the transition amplitude from the initial point y into the final one x, and the integral of the motion $\hat{\mathbf{X}}$ as the operator of the initial coordinate. The equation (2.7a, b) determine the Green function to within a constant factor depending only on time. To find this factor one should take into account the Schrödinger equation (2.2).

In the following Sections we shall consider in detail two cases when equation (2.3) can be solved exactly. However, even if the exact expressions for the operators $\hat{\mathbf{X}}$ and $\hat{\mathbf{P}}$ cannot be obtained, one can write approximate expressions for these operators using the Beiker-Hausdorff formula

$$\hat{\mathbf{X}} = \exp\left(-\frac{it}{\hbar}\hat{H}\right)\hat{\mathbf{x}}\exp\left(\frac{it}{\hbar}\hat{H}\right) = \hat{\mathbf{x}} + \left(\frac{t}{i\hbar}\right)[\hat{H}, \hat{\mathbf{x}}] + \frac{1}{2!}\left(\frac{t}{i\hbar}\right)^2[\hat{H}, [\hat{H}, \hat{\mathbf{x}}]] + \ldots$$

or some others of the same type. Then one can obtain approximate expressions for the Green function. Such procedure may prove to be especially useful for approximate calculations of the equilibrium density matrix (it can be obtained from the Green function by means of the substitution $t = -i\hbar/\kappa T$, where T is the absolute temperature and κ is the Bolzmann constant).

Equations (2.7a, b) are not symmetrical with respect to the variables x and y. But one can write instead of equation (2.7b) another one, which is similar to equation (2.7a)

$$\hat{\mathbf{Y}}_{\{2\}}G(\mathbf{x}, \mathbf{y}; t) = \mathbf{x}G(\mathbf{x}, \mathbf{y}; t) \tag{2.8}$$

$$\hat{\mathbf{Y}} = (\hat{U}^{-1}\hat{\mathbf{x}}\hat{U})^T \tag{2.9}$$

The proof is analogous to that of equation (2.7), if one takes into account that $G(\mathrm{x}, \mathrm{y}; t) = \hat{U}^T_{\{2\}}\delta(\mathrm{x} - \mathrm{y})$. The system of equations (2.7a) and (2.8) is symmetrical with respect to x and y. If the operator x can be expressed as a function of the operators $\hat{\mathbf{X}}$ and $\hat{\mathbf{P}}$: $\hat{\mathrm{x}} = \hat{U}^{-1}\hat{\mathbf{X}}\hat{U} = f(\hat{\mathbf{X}}, \hat{\mathbf{P}})$, – then comparing this relation with equation (2.9) one can write the following expression for the operator $\hat{\mathbf{Y}}$ as the function of the operators $\hat{\mathrm{x}}$ and $\hat{\mathbf{P}}$: $\hat{\mathbf{Y}} = [f(\hat{\mathrm{x}}, \hat{\mathrm{p}})]^T$. Although at the initial moment the operator $\hat{\mathbf{Y}}$ coincides with the operator $\hat{\mathrm{x}}$, it is not the integral of the motion for the system with the Hamiltonian $\hat{H}$, since $\hat{\mathbf{Y}}$ satisfies not equation (2.2) but the equation

$$i\hbar\frac{\partial}{\partial t}\hat{\mathbf{Y}} = [\hat{\tilde{H}}, \hat{\mathbf{Y}}]; \qquad \hat{\tilde{H}} = (\hat{U}^{-1}\hat{H}\hat{U})^T \tag{2.10}$$

If the Hamiltonian does not depend on time, the operator $\hat{U}$ commutes with $\hat{H}$, because in this case $\hat{U} = \exp(-(it/\hbar)\hat{H})$. Then $\hat{\tilde{H}} = \hat{H}^T$. This means that if the Hamiltonian is stationary and symmetrical ($\hat{H} = \hat{H}^T$), then the Green function is symmetrical with respect to its arguments x and y, due to the relation $\hat{\mathbf{X}} = \hat{\mathbf{Y}}$. If the Hamiltonian is only stationary, then equation (2.3) and (2.10) lead to the relation $\hat{\mathbf{Y}}(t) = \hat{\mathbf{X}}^T(-t)$. Therefore in the case of stationary Hamiltonians it is sufficiently as a rule to solve only equation (2.7a) which determines the dependence of the Green function on the first variable x; after this the dependence on the second variable y can be established without solving differential equations.

For example let us consider the case when the operator $\hat{\mathbf{X}}$ can be expressed as a function of the operators $\hat{\mathrm{x}}$ and $\hat{\mathrm{p}}$, all products $\hat{\mathrm{x}}\hat{\mathrm{p}}$ being symmetrized. Suppose the solution to equation (2.7a) to be analytical with respect to coefficients of this function (these coefficients depend on time). Then the dependence of the Green function on the second argument can be derived from the known dependence on the first argument simply by analytical continuation, if one makes in all coefficients the substitution $t \to -t$ and changes signs of the coefficients standing before odd powers of the operators $\hat{p}_\kappa$ in the expression for $\hat{\mathbf{X}}$ (the transposition transforms the operator $\hat{p}_\kappa$ into $-\hat{p}_\kappa$). These reasonings will be illustrated in the next Section.

Sometimes it may be more convenient to use not operators $\hat{Q}_j$ but some functions of them $\varphi_\kappa(\hat{Q}_j)$, $\kappa, j = 1, 2, \ldots 2N$ (These functions are of course integrals of the motion too.) Then equations (2.7a, b) should be replaced by the following ones

$$\varphi_\kappa(\hat{Q})_{\{1\}}G(\mathrm{x}, \mathrm{y}; t) = [\varphi_\kappa(\hat{\mathrm{q}})]^T_{\{2\}}G(\mathrm{x}, \mathrm{y}; t)$$
$$\kappa = 1, 2, \ldots 2N \tag{2.11}$$

Equation (2.8) should be changed analogously.

Up to now we considered only the coordinate representation. However equations similar to equations (2.7a, b) can be written in an arbitrary representation according to the following scheme. Let $2N$ independent operators

$\hat{d}_j$ exist, and their action on the first argument of the Green function at the initial moment is equivalent to the action of some operators $\hat{\varphi}(\hat{d}_j)$ on the second argument

$$\hat{\mathbf{d}}_{\{1\}}G(\mathbf{x}, \mathbf{y}; 0) = \hat{\varphi}(\hat{\mathbf{d}})_{\{2\}}G(\mathbf{x}, \mathbf{y}; 0) \tag{2.12}$$

(here x and y are arbitrary variables on which the Green function depends; their physical meaning depends on the representation chosen). If the integral of the motion $\hat{\mathbf{D}} = \hat{U}\hat{\mathbf{d}}\hat{U}^{-1}$ is known, the Green function is the solution to the equation

$$\hat{\mathbf{D}}_{\{1\}}G(\mathbf{x}, \mathbf{y}; t) = \hat{\varphi}(\hat{\mathbf{d}})_{\{2\}}G(\mathbf{x}, \mathbf{y}; t) \tag{2.13}$$

For example, in the momentum representation one obtains again equation (2.7a, b) in which one should make the substitution $\mathbf{x} \rightleftarrows \mathbf{p}$ and take into account that in this representation

$$\hat{\mathbf{p}}^T = \hat{\mathbf{p}}; \qquad \hat{\mathbf{x}}^T = -\hat{\mathbf{x}}; \qquad \hat{\mathbf{x}} = i\hbar\, \partial/\partial \mathbf{p}.$$

To illustrate the general scheme let us consider the representation occurring very useful for solving many physical problems, namely the coherent states representation (Glauber, 1963; Bargmann, 1961). In this representation every ket-vector $|f\rangle$ is represented by the entire analytic function of a complex argument $f(\boldsymbol{\alpha}^*) = \exp(\frac{1}{2}|\boldsymbol{\alpha}|^2)\langle\boldsymbol{\alpha}|f\rangle$ (the asterisk means the complex conjugation), where $|\boldsymbol{\alpha}\rangle$ is the eigenstate of nonhermitian operators $\hat{a}_j$ satisfying the commutation relations of boson creation and annihilation operators:

$$\begin{aligned} &\hat{a}_j|\boldsymbol{\alpha}\rangle = \boldsymbol{\alpha}_j|\boldsymbol{\alpha}\rangle; \qquad \boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots \alpha_N); \qquad j = 1, 2, \ldots N \\ &[\hat{a}_j, \hat{a}_\kappa] = 0; \qquad [\hat{a}_j, \hat{a}_\kappa^+] = \delta_{j\kappa}; \qquad \hat{\mathbf{a}} = (\hat{a}_1, \hat{a}_2, \ldots \hat{a}_N) \end{aligned} \tag{2.14}$$

It is well known that the spectrum of operators $\hat{a}_j$ consists of all complex numbers. The operators $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^+$ act on the functions $f(\boldsymbol{\alpha}^*)$ according to the formulae

$$\hat{\mathbf{a}}f(\boldsymbol{\alpha}^*) = \frac{\partial}{\partial \boldsymbol{\alpha}^*} f(\boldsymbol{\alpha}^*); \qquad \hat{\mathbf{a}}^+ f(\boldsymbol{\alpha}^*) = \boldsymbol{\alpha}^* f(\boldsymbol{\alpha}^*) \tag{2.15}$$

and the kernel of the unity operator is $G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; 0) = \exp(\boldsymbol{\alpha}^*\boldsymbol{\beta})$:

$$\begin{aligned} f(\boldsymbol{\alpha}^*) &= \int \exp(\boldsymbol{\alpha}^*\boldsymbol{\beta}) f(\boldsymbol{\beta}^*)\, d\mu(\boldsymbol{\beta}); \\ d\mu(\boldsymbol{\beta}) &= \pi^{-N} \exp(-|\boldsymbol{\beta}|^2)\, dRe\boldsymbol{\beta}\, d\,\mathrm{Im}\boldsymbol{\beta} \end{aligned} \tag{2.16}$$

Therefore equation (2.12) can be written as follows:

$$\begin{aligned} &\hat{\mathbf{a}}_{\{1\}}G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; 0) = \boldsymbol{\beta} e^{\boldsymbol{\alpha}^*\boldsymbol{\beta}} = \hat{\mathbf{a}}^T_{\{2\}}G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; 0); \\ &\hat{\mathbf{a}}^+_{\{1\}}G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; 0) = \frac{\partial}{\partial \boldsymbol{\beta}} e^{\boldsymbol{\alpha}^*\boldsymbol{\beta}} = (\hat{\mathbf{a}}^+)^T_{\{2\}}G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; 0) \end{aligned} \tag{2.17}$$

(Note that one should act on the second argument $\boldsymbol{\beta}$ not by the operators $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^+$, but by the complex conjugate operators $\hat{\mathbf{a}}^* = (\hat{\mathbf{a}}^+)^T$ and $(\hat{\mathbf{a}}^+)^* = \hat{\mathbf{a}}^T$. Introducing the integrals of the motion $\hat{\mathbf{A}} = \hat{U}\hat{\mathbf{a}}\hat{U}^{-1}$ and $\hat{\mathbf{F}} = \hat{U}\hat{\mathbf{a}}^+\hat{U}^{-1}$ (if the Hamiltonian is nonhermitian, $\hat{\mathbf{F}} \neq \hat{\mathbf{A}}^+$; nevertheless the relations $[\hat{A}_j, \hat{F}_\kappa] = \delta_{j\kappa}$ always hold), one obtains the equations for the Green function:

$$\hat{\mathbf{A}}_{\{1\}} G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; t) = \boldsymbol{\beta} G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; t)$$

$$\hat{\mathbf{F}}_{\{1\}} G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; t) = \frac{\partial}{\partial \boldsymbol{\beta}} G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; t) \tag{2.18}$$

The physical significance of the function $G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; t)$ is that the value $G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; t) \exp[-\frac{1}{2}(|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2)]$ is the transition amplitude from the initial coherent state $|\boldsymbol{\beta}, 0\rangle$ into the final one $|\boldsymbol{\alpha}, t\rangle$. One of the main advantages of the coherent states representation is that the Green function in this representation has no singularities (it is an entire analytical function). The transformation to the coordinate representation can be performed by the formula

$$G(\mathbf{x}, \mathbf{y}) = \int \langle \mathbf{x} | \boldsymbol{\alpha} \rangle G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}) \langle \boldsymbol{\beta}^* | \mathbf{y} \rangle \times \exp[\tfrac{1}{2}(|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2)] \, d\mu(\boldsymbol{\alpha}) \, d\mu(\boldsymbol{\beta}) \tag{2.19}$$

where $\langle \mathbf{x} | \boldsymbol{\alpha} \rangle$ is the coordinate wave function corresponding to the ket-vector $|\boldsymbol{\alpha}\rangle$.

3. *The General Quadratic System*

The Hamiltonian of the most general N-dimensional quadratic system can be written as follows,

$$\hat{H} = \tfrac{1}{2}\hat{\mathbf{q}}B(t)\hat{\mathbf{q}} + \mathbf{C}(t)\hat{\mathbf{q}} + \Phi(t);$$

$$\mathbf{q} = \begin{pmatrix} \mathbf{p} \\ \mathbf{x} \end{pmatrix}; \qquad \mathbf{C} = \begin{pmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \end{pmatrix}; \qquad B = \begin{Vmatrix} b_1 & b_2 \\ b_3 & b_4 \end{Vmatrix} \tag{3.1}$$

N-dimensional vectors c_1 and c_2 and N-dimensional matrices $b_j, j = 1, 2, 3, 4$, may be arbitrary complex functions of time. Evidently, one can always believe the matrix B to be symmetrical. One can easily verify that the $2N$-dimensional vector

$$\hat{\mathbf{Q}} = \begin{pmatrix} \hat{\mathbf{P}} \\ \hat{\mathbf{X}} \end{pmatrix} = \Lambda(t)\hat{\mathbf{q}} + \boldsymbol{\Delta}(t); \qquad \Lambda = \begin{Vmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{Vmatrix}; \qquad \boldsymbol{\Delta} = \begin{pmatrix} \boldsymbol{\delta}_1 \\ \boldsymbol{\delta}_2 \end{pmatrix} \tag{3.2}$$

is the integral of the motion coinciding with $\hat{\mathbf{q}}$ at the initial moment, provided the $2N$-dimensional matrix $\Lambda(t)$ and vector $\boldsymbol{\Delta}(t)$ satisfy the following system of ordinary differential equations of the first order:

$$\dot{\Lambda} = \Lambda\Sigma B; \qquad \Lambda(0) = E_{2N}; \qquad \Sigma = \begin{Vmatrix} 0 & E_N \\ -E_N & 0 \end{Vmatrix}$$
$$\dot{\boldsymbol{\Delta}} = \Lambda\Sigma\mathbf{C}; \qquad \boldsymbol{\Delta}(0) = 0; \tag{2.3}$$

E_N is the $N \times N$ unity matrix. It is well known that the solution to the system (3.3) always exists, and it is unique. Further we shall show that the Green function is completely determined by the matrices $\lambda_j, j = 1, 2, 3, 4$ and vectors $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$. Since the commutator of integrals of the motion is the integral of the motion too, the following relations are valid: $[\hat{Q}_j, \hat{Q}_\kappa] = [\hat{q}_j, \hat{q}_\kappa] = -i\hbar\Sigma_{j\kappa}$. Therefore the matrix Λ must satisfy the relation $\Lambda\Sigma\tilde{\Lambda} = \Sigma$, i.e. it must be symplectic (see also Berezin, 1966; Moshinsky and Quesne, 1971). The wave means a transposed matrix. The symplectic condition imposes the following restrictions on the elements of the matrix Λ (these elements are generally speaking complex, except the special case when the Hamiltonian is hermitian and consequently the matrix B is real):

$$\begin{aligned} &\lambda_2\tilde{\lambda}_1 = \lambda_1\tilde{\lambda}_2 \qquad \lambda_4\tilde{\lambda}_1 - \lambda_3\tilde{\lambda}_2 = E_N \\ &\lambda_3\tilde{\lambda}_4 = \lambda_4\tilde{\lambda}_3 \qquad \tilde{\lambda}_4\lambda_1 - \tilde{\lambda}_2\lambda_3 = E_N \\ &\tilde{\lambda}_4\lambda_2 = \tilde{\lambda}_2\lambda_4 \qquad \Lambda^{-1} = \begin{Vmatrix} \tilde{\lambda}_4 & -\tilde{\lambda}_2 \\ -\tilde{\lambda}_3 & \tilde{\lambda}_1 \end{Vmatrix}; \qquad \det\Lambda = 1 \\ &\tilde{\lambda}_1\lambda_3 = \tilde{\lambda}_3\lambda_1 \end{aligned} \tag{3.4}$$

The Green function satisfies the equations (see equations (2.7a, b) and (3.2))

$$\left(-i\hbar\lambda_3\frac{\partial}{\partial \mathrm{x}} + \lambda_4\mathrm{x} + \boldsymbol{\delta}_2\right)G(\mathrm{x}, \mathrm{y}; t) = \mathrm{y}G(\mathrm{x}, \mathrm{y}; t)$$

$$\left(-i\hbar\lambda_1\frac{\partial}{\partial \mathrm{x}} + \lambda_2\mathrm{x} + \delta_1\right)G(\mathrm{x}, \mathrm{y}; t) = i\hbar\frac{\partial}{\partial \mathrm{y}}G(\mathrm{x}, \mathrm{y}; t)$$

If $\det\lambda_3 \neq 0$, these equations can be easily integrated, so that

$$G(\mathrm{x}, \mathrm{y}; t) = \exp\left\{\varphi(t) - \frac{i}{2\hbar}[\mathrm{x}\lambda_3^{-1}\lambda_4\mathrm{x} - 2\mathrm{x}\lambda_3^{-1}\mathrm{y} + \right.$$
$$\left. + \mathrm{y}\lambda_1\lambda_3^{-1}\mathrm{y} + 2\mathrm{x}\lambda_3^{-1}\boldsymbol{\delta}_2 + 2\mathrm{y}(\boldsymbol{\delta}_1 - \lambda_1\lambda_3^{-1}\boldsymbol{\delta}_2)]\right\}$$

Note that the matrices $\lambda_3^{-1}\lambda_4$ and $\lambda_1\lambda_3^{-1}$ are symmetrical, as well as any matrix μ which may occur further in a quadratic form of the type $z\mu z$. This can be easily proved in every case with the aid of equation (3.4). To find the phase $\varphi(t)$ one should substitute $G(\mathrm{x}, \mathrm{y}; t)$ into the Schrödinger equation. Then the following relation can be obtained:

$$\dot{\varphi} = \tfrac{1}{2}Tr(\lambda_3^{-1}\lambda_4 b_1 - b_3) + \frac{i}{\hbar}\mathrm{c}_1\lambda_3^{-1}\delta_2 - \frac{i}{2\hbar}\delta_2\tilde{\lambda}_3^{-1}b_1\lambda_3^{-1}\delta_2 - \frac{i}{\hbar}\Phi$$

(there is a misprint in the similar expression in the paper by Malkin *et al.* (1973), equation (18)). This equation can be easily integrated, if one takes into account the relations following equation (3.3): $\lambda_4 b_1 = -\dot{\lambda}_3 + \lambda_3 b_3$; $\dot{\boldsymbol{\delta}}_2 = \lambda_3\mathrm{c}_2 - \lambda_4\mathrm{c}_1$ and the identity $Tr(\dot{\lambda}\lambda^{-1}) = d/dt \ln\det\lambda(t)$. The time-independent constant is determined by the requirement $\lim_{t\to+0} G(\mathrm{x}, \mathrm{y}; t) = \delta(\mathrm{x} - \mathrm{y})$. Having

performed all calculations one can obtain the following expression for the Green function:

$$G(\mathrm{x}, \mathrm{y}; t) = (-2\pi i\ )^{-N/2}(\det \lambda_3)^{-1/2} \exp\left\{-\frac{i}{2\hbar}\left[\mathrm{x}\lambda_3^{-1}\lambda_4\mathrm{x} - \right.\right.$$
$$- 2\mathrm{x}\lambda_3^{-1}\mathrm{y} + \mathrm{y}\lambda_1\lambda_3^{-1}\mathrm{y} + 2\mathrm{x}\lambda_3^{-1}\delta_2 +$$
$$\left.\left. + 2\mathrm{y}(\delta_1 - \lambda_1\lambda_3^{-1}\delta_2) + \delta_2\lambda_1\lambda_3^{-1}\delta_2 - 2\int_0^t (\dot{\delta}_1\delta_2 - \Phi)\, d\tau\right]\right\} \quad (3.5a)$$

If the matrix Λ and the vector $\boldsymbol{\Delta}$ are real, then the function $G(\mathrm{x}, \mathrm{y})$ (3.5a) can be considered as the kernel of the unitary transformation corresponding to the linear canonical transformation of operators $\hat{\mathrm{x}}$ and $\hat{\mathrm{p}}$ (3.2). In this interpretation formula analogous to (3.5a) were obtained (to within a phase factor) by Moshinsky and Quesne (1971) (when $\Delta = 0$) and Boon and Seligman (1973) ($\Delta \neq 0$).

We want to write also formulae for the Green function in the momentum representation $G(\mathrm{p}_2, \mathrm{p}_1; t)$ and for the functions $G(\mathrm{p}_2, \mathrm{x}_1; t)$ and $G(\mathrm{x}_2, \mathrm{p}_1; t)$ – the transition amplitudes form the initial coordinate eigenstate $|\mathrm{x}_1; 0\rangle$ into the final momentum eigenstate $|\mathrm{p}_2; t\rangle$ and vice versa. All these functions can be obtained either with the aid of equations (2.13), (2.7) and (3.2) or by means of the Fourier transformation of equation (3.5a).

$$G(\mathrm{p}_2, \mathrm{p}_1; t) = (2\pi i\hbar)^{-N/2}(\det \lambda_2)^{-1/2} \exp\left\{\frac{i}{2\hbar}\left[\mathrm{p}_2\lambda_2^{-1}\lambda_1\mathrm{p}_2 - \right.\right.$$
$$- 2\mathrm{p}_2\lambda_2^{-1}\mathrm{p}_1 + \mathrm{p}_1\lambda_4\lambda_2^{-1}\mathrm{p}_1 + 2\mathrm{p}_2\lambda_2^{-1}\delta_1 + 2\mathrm{p}_1(\delta_2 - \lambda_4\lambda_2^{-1}\delta_1) +$$
$$\left.\left. + \delta_1\lambda_4\lambda_2^{-1}\delta_1 - 2\int_0^t (\dot{\delta}_2\delta_1 + \Phi)\, d\tau\right]\right\} \quad (3.5b)$$

$$G(\mathrm{x}_2, \mathrm{p}_1; t) = (2\pi\hbar)^{-N/2}(\det \lambda_1)^{-1/2} \exp\left\{-\frac{i}{2\hbar}\left[\mathrm{x}_2\lambda_1^{-1}\lambda_2\mathrm{x}_2 - \right.\right.$$
$$- 2\mathrm{x}_2\lambda_1^{-1}\mathrm{p}_1 - \mathrm{p}_1\lambda_3\lambda_1^{-1}\mathrm{p}_1 + 2\mathrm{x}_2\lambda_1^{-1}\delta_1 - 2\mathrm{p}_1(\delta_2 - \lambda_3\lambda_1^{-1}\delta_1) -$$
$$\left.\left. - \delta_1\lambda_3\lambda_1^{-1}\delta_1 + 2\int_0^t (\dot{\delta}_2\delta_1 + \Phi)\, d\tau\right]\right\} \quad (3.5c)$$

$$G(\mathrm{p}_2, \mathrm{x}_1; t) = (2\pi\hbar)^{-N/2}(\det \lambda_4)^{-1/2} \exp\left\{\frac{i}{2\hbar}\left[\mathrm{p}_2\lambda_4^{-1}\lambda_3\mathrm{p}_2 - \right.\right.$$
$$- 2\mathrm{p}_2\lambda_4^{-1}\mathrm{x}_1 - \mathrm{x}_1\lambda_2\lambda_4^{-1}\mathrm{x}_1 + 2\mathrm{p}_2\lambda_4^{-1}\delta_2 - 2\mathrm{x}_1(\delta_1 - \lambda_2\lambda_4^{-1}\delta_2) -$$
$$\left.\left. - \delta_2\lambda_2\lambda_4^{-1}\delta_2 + 2\int_0^t (\dot{\delta}_1\delta_2 - \Phi)\, d\tau\right]\right\} \quad (3.5d)$$

Let us consider the case when the Hamiltonian does not depend on time. Then the expression for the Green function can be simplified. Since the operator $\hat{\mathrm{X}}$ equals $\lambda_3\hat{\mathrm{p}} + \lambda_4\hat{\mathrm{x}} + \boldsymbol{\delta}_2$, the operator $\hat{\mathrm{Y}}$ must be equal to $\hat{\mathrm{X}}^T(-t) =$

$-\lambda_3(-t)\hat{\mathbf{p}} + \boldsymbol{\delta}_2(-t) + \lambda_4(-t)\hat{\mathbf{x}}$. Besides, the argument of the exponent in the right-hand part of equation (3.5a) is the analytical function with respect to the coefficients λ_3, λ_4 and $\boldsymbol{\delta}_2$. Consequently, applying the general rule formulated in the previous Section one can write the following formula:

$$G(\mathbf{x}, \mathbf{y}; t) = (-2\pi i\hbar)^{-N/2} [\det \lambda_3(t)]^{-1/2} \exp\left\{-\frac{i}{2\hbar}\left[\mathbf{x}\lambda_3^{-1}(t)\lambda_4(t)\mathbf{x} - \right.\right.$$

$$-2\mathbf{x}\lambda_3^{-1}(t)\mathbf{y} - \mathbf{y}\lambda_3^{-1}(-t)\lambda_4(-t)\mathbf{y} + 2\mathbf{x}\lambda_3^{-1}(t)\boldsymbol{\delta}_2(t) - 2\mathbf{y}\lambda_3^{-1}(-t)\boldsymbol{\delta}_2(-t) +$$

$$\left.\left. + \boldsymbol{\delta}_2(t)\lambda_1(t)\lambda_3^{-1}(t)\boldsymbol{\delta}_2(t) - 2\int_0^t (\dot{\boldsymbol{\delta}}_1\boldsymbol{\delta}_2 - \Phi)\, d\tau\right]\right\} \tag{3.6}$$

Therefore in the stationary case it is sufficient to know only two matrices λ_3 and λ_4 to obtain the Green function. Comparing equations (3.5a) and (3.6) one can come to a conclusion that the following relations must hold in the stationary case

$$\tilde{\lambda}_1(-t) = \lambda_4(t); \qquad \tilde{\lambda}_3(-t) = -\lambda_3(t); \qquad \tilde{\lambda}_2(-t) = -\lambda_2(t);$$

$$\boldsymbol{\delta}_1(t) = \tilde{\lambda}_3^{-1}(t)\boldsymbol{\delta}_2(-t) - \lambda_3^{-1}(-t)\lambda_4(-t)\boldsymbol{\delta}_2(t);$$

$$\boldsymbol{\delta}_2(t) = \lambda_4(t)\lambda_2^{-1}(t)\boldsymbol{\delta}_1(t) - \tilde{\lambda}_2^{-1}(t)\boldsymbol{\delta}_1(-t) \tag{3.7}$$

One can verify that these relations are really the consequence of equations (3.3) (one of the possible ways to check these identities is to develop the solutions to equation (3.3) into power series with respect to the variable t).

It is interesting to compare two systems: one described by the Hamiltonian $\hat{H}_0(t) = \frac{1}{2}\hat{\mathbf{q}}B(t)\hat{\mathbf{q}}$ and another having the Hamiltonian $\hat{H}(t) = \hat{H}_0(t) + \mathbf{C}(t)\hat{\mathbf{q}}$. It can be shown that the corresponding evolution operators are related as follows:

$$\hat{U}(t_2, t_1) = \hat{S}^{-1}(t_2)\hat{U}_0(t_2, t_1)\hat{S}(t_1) \times \exp\left(\frac{i}{2\hbar}\int_{t_1}^{t_2} \mathbf{C}\mathbf{f}\, d\tau\right) \tag{3.8}$$

where the $2N$-dimensional vector $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2)$ is an arbitrary solution to the equation

$$\dot{\mathbf{f}} = -\Sigma B\mathbf{f} + \Sigma\mathbf{C} \tag{3.8a}$$

and the unitary displacement operator $\hat{S}(t)$ and its kernel $S(\mathbf{x}_2, \mathbf{x}_1; t)$ are defined as follows

$$\hat{S}(t) = \exp\left(\frac{i}{\hbar}\mathbf{f}\,\Sigma\,\hat{\mathbf{q}}\right);$$

$$S(\mathbf{x}_2, \mathbf{x}_1; t) = \delta(\mathbf{x}_2 - \mathbf{x}_1 - \mathbf{f}_2)\exp\left(\frac{i}{2\hbar}[2\mathbf{f}_1\mathbf{x}_2 - \mathbf{f}_1\mathbf{f}_2]\right) \tag{3.8b}$$

The origin of the last factor in the right-hand part of equation (3.8) can be understood, if one takes into account that the operator $\hat{S}(t)$ transforms the operator $\hat{H}_0$ into the operator $\hat{H}(t)$ only to within a certain term $\phi(t)$ $\hat{S}^{-1}(t)\hat{H}_0(t)\hat{S}(t) = \hat{H}(t) + \phi(t)$. The last exponent does eliminate this additional term. Just this factor leads after all to the appearance of the integral $\int_0^t \dot{\boldsymbol{\delta}}_1 \boldsymbol{\delta}_2 \, d\tau$ in the right-hand part of equation (3.5a). By the way, if the matrix B is nonsingular, and the Hamiltonian $\hat{H}$ does not depend on time, equation (3.8a) has the stationary solution $\mathbf{f} = B^{-1}\mathbf{C}$. Therefore the following important formula is valid in this case

$$\mathrm{Tr}\hat{U}(t) = \mathrm{Tr}\hat{U}_0(t) \exp\left(\frac{it}{2\hbar}\mathbf{C}B^{-1}\mathbf{C}\right) \tag{3.9}$$

The formula (3.5a) is valid only if the matrix λ_3 is nonsingular. If $\det \lambda_3 = 0$, but $\det \lambda_4 \neq 0$, the Green function in the coordinate representation can be obtained from equation (3.5d) with the aid of the Fourier transformation.

Let us assume for the sake of simplicity the Hamiltonian to be hermitian. Then the matrix Λ is real. Since the matrix $\lambda_4^{-1}\lambda_3$ is symmetrical, a certain orthogonal transformation S reducing this matrix to a diagonal form exists:

$$\lambda_4^{-1}\lambda_3 = S^{-1} \, \mathrm{diag}(\mu_1, \mu_2, \ldots \mu_\kappa, 0, \ldots 0)S$$
$$\mu_j \neq 0, \qquad j \leqslant \kappa$$

Then the result of the calculation of the Fourier integral can be represented as follows

$$G(\mathbf{x}_2, \mathbf{x}_1; t) = (-2\pi i\hbar)^{-\kappa/2}(\det \lambda_4)^{-1/2} \prod_{j=1}^{\kappa} \mu_j^{-1/2} \exp\left(\frac{-iz_j^2}{2\hbar\mu_j}\right) \times$$
$$\times \prod_{j=\kappa+1}^{N} \delta(z_j) \exp\left\{-\frac{i}{2\hbar}\left[\mathbf{x}_1\lambda_2\lambda_4^{-1}\mathbf{x}_1 + 2\mathbf{x}_1(\boldsymbol{\delta}_1 - \lambda_2\lambda_4^{-1}\boldsymbol{\delta}_2) + \right.\right.$$
$$\left.\left. + \boldsymbol{\delta}_2\lambda_2\lambda_4^{-1}\boldsymbol{\delta}_2 - 2\int_0^t (\dot{\boldsymbol{\delta}}_1\boldsymbol{\delta}_2 - \Phi)\, d\tau\right]\right\};$$
$$\mathbf{z} = S(\mathbf{x}_2 - \lambda_4^{-1}\mathbf{x}_1 + \lambda_4^{-1}\boldsymbol{\delta}_2) \tag{3.10}$$

If $\det \lambda_4 = 0$, but $\det \lambda_2 \neq 0$ or $\det \lambda_1 \neq 0$, formula analogous to equation (3.10) can be derived from equations (3.5b) or (3.5c). When all four λ-matrices become singular, one of the possible ways to obtain the Green function in the coordinate representation is to derive it from the Green function in the coherent states representation, since the latter one always has no singularities. Thus we are proceeding to calculating the Green function in the coherent states representation.

Hereafter we believe the Hamiltonian to be hermitian. This assumption is not fundamental, but it enables to simplify formula. An arbitrary hermitian quadratic Hamiltonian can be represented as

$$\hat{H} = \tfrac{1}{2}[\hat{a}^+ d_0 \hat{a} + \hat{a}\, d_0^* \hat{a}^+ + \hat{a}\, d_1 \hat{a} + \hat{a}^+ d_1^* \hat{a}^+] + \mathbf{f}\hat{a} + \mathbf{f}^* \hat{a}^+ + \Phi(t) \quad (3.11)$$

where the $N \times N$-matrix d_1 is symmetrical, and d_0 is a hermitian matrix. The integral of the motion $\hat{A} = \hat{U}\hat{a}\hat{U}^{-1}$ can be expressed as

$$\hat{\mathbf{A}} = \xi(t)\hat{\mathbf{a}} + \eta(t)\hat{\mathbf{a}}^+ + \gamma(t) \quad (3.12a)$$

if the N-dimensional complex matrices ξ and η and the vector γ satisfy the equations

$$\begin{aligned} i\hbar\dot{\xi} &= -\xi d_0 + \eta d_1; \qquad \xi(0) = E_N \\ i\hbar\dot{\eta} &= -\xi d_1^* + \eta d_0^*; \qquad \eta(0) = 0 \\ i\hbar\dot{\gamma} &= -\xi \mathbf{f}^* + \eta \mathbf{f}; \qquad \gamma(0) = 0 \end{aligned} \quad (3.12b)$$

Since the commutation relations between the operators $\hat{A}_j$ and $\hat{A}_\kappa$ must be the same as between $\hat{a}_j$ and $\hat{a}_\kappa$, the following identities must be valid (one can verify that these identities follow equation (3.12b))

$$\begin{aligned} \xi\xi^+ - \eta\eta^+ &= E_N; \qquad \xi\tilde{\eta} = \eta\tilde{\xi} \\ \xi^+\xi - \tilde{\eta}\eta^* &= E_N; \qquad \eta^+\xi = \tilde{\xi}\eta^* \end{aligned} \quad (3.13)$$

The first of these identities leads to the relation $(\mathbf{z}, \xi\xi^+\mathbf{z}) = (\mathbf{z}, \mathbf{z}) + (\eta^+\mathbf{z}, \eta^+\mathbf{z}) \geqslant (\mathbf{z}, \mathbf{z})$, $\mathbf{z}$ being an arbitrary complex vector. Consequently, the matrix ξ is nonsingular, because otherwise such a vector $\mathbf{z} \neq 0$ would exist that $\xi^+\mathbf{z} = 0$. For this reason the Green function in the coherent states representation has no singularities. Equations (2.18) and (3.12)–(3.13) lead to the following expression for the Green function

$$\begin{aligned} G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; t) = (\det \xi)^{-1/2} \exp\Big\{ & -\tfrac{1}{2}\boldsymbol{\alpha}^*\xi^{-1}\eta\boldsymbol{\alpha}^* + \boldsymbol{\alpha}^*\xi^{-1}\boldsymbol{\beta} + \\ & + \tfrac{1}{2}\boldsymbol{\beta}\eta^*\xi^{-1}\boldsymbol{\beta} - \boldsymbol{\alpha}^*\xi^{-1}\gamma + \boldsymbol{\beta}(\gamma^* - \eta^*\xi^{-1}\gamma) + \\ & + \tfrac{1}{2}\gamma\eta^*\xi^{-1}\gamma - \int_0^t [\dot{\gamma}^*\gamma + i\Phi]\, d\tau \Big\} \end{aligned} \quad (3.14)$$

Using this formula and equation (2.19) one can obtain the Green function in the coordinate representation in the case when all four λ-matrices are degenerate. We do not write the corresponding formula because it is very cumbersome.

The kernel of the unitary transformation corresponding to the canonical transformation of the operators $\hat{\mathbf{a}}$ and $\hat{\mathbf{a}}^+$ (3.12a) was calculated by Berezin (1966) both for the Fermi and Bose operators. His result coincides with equation (3.14) except the phase factor.

4. *The Nonstationary Singular Oscillator*

As the second example we consider the nonquadratic one-dimensional system with the Hamiltonian

$$\hat{H} = \tfrac{1}{2}\hat{p}^2 + gx^{-2} + \tfrac{1}{2}\omega^2(t)x^2; \qquad x > 0; \qquad g > -\tfrac{1}{8}; \qquad \equiv 1 \tag{4.1}$$

Let us introduce the following operators:

$$\hat{q}_1 = \hat{x}^2; \qquad \hat{q}_2 = \hat{p}^2 + 2g\hat{x}^{-2}; \qquad \hat{q}_3 = \hat{x}\hat{p} + \hat{p}\hat{x} \tag{4.2}$$

One can verify that the integrals of the motion $\hat{Q}_j = \hat{U}\hat{q}_j\hat{U}^{-1}$ are linear functions of the operators $\hat{q}_j$

$$\begin{pmatrix} \hat{Q}_1 \\ \hat{Q}_2 \\ \hat{Q}_3 \end{pmatrix} = \begin{pmatrix} \dot{\mu}^2 & \mu^2 & -\mu\dot{\mu} \\ \dot{\lambda}^2 & \lambda^2 & -\lambda\dot{\lambda} \\ -2\dot{\lambda}\dot{\mu} & -2\lambda\mu & \lambda\dot{\mu} + \mu\dot{\lambda} \end{pmatrix} \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \\ \hat{q}_3 \end{pmatrix} \tag{4.3}$$

The functions $\lambda(t)$ and $\mu(t)$ satisfy the equation

$$\ddot{z} + \omega^2(t)z = 0 \tag{4.4a}$$

and the initial conditions

$$\lambda(0) = \dot{\mu}(0) = 1; \qquad \dot{\lambda}(0) = \mu(0) = 0 \tag{4.4b}$$

The Green function is the solution to the equation of the type (2.11)

$$\hat{Q}_j\{_1\}G(x_2, x_1; t) = \hat{q}_j^T\{_2\}G(x_2, x_1; t) \tag{4.5}$$

The only solution to this equation turning into zero when $x_1 = x_2 = 0$ and coinciding with the delta-function when $t = 0$ is

$$G(x_2, x_1; t) = \mu^{-1}(x_1x_2)^{1/2}J_b(x_1x_2\mu^{-1}) \times$$
$$\times \exp\left\{-\frac{i}{2}(b+1)\pi + \frac{i}{2\mu}(\dot{\mu}x_2^2 + \lambda x_1^2)\right\};$$
$$b = \tfrac{1}{2}(1 + 8g)^{1/2} \tag{4.6}$$

J_b is the Bessel function.

It is interesting to consider another representation, which has much in common with the usual coherent states one. In this representation (let us call it the z-representation) each ket-vector $|\psi\rangle$ is represented by the complex

function of the complex argument $\psi(z^*) = \langle z|\psi\rangle$, the state $|z\rangle$ being the eigenstate of the nonhermitian operator

$$\hat{A} = \frac{1}{2\omega_0}[(\hat{p} - i\omega_0 x)^2 + 2g/x^2] = \frac{1}{2\omega_0}[\hat{q}_2 - \omega_0{}^2 q_1 - i\omega_0 \hat{q}_3];$$

$$\omega_0 = \omega(t \to -\infty); \qquad \hat{A}|z\rangle = z|z\rangle \tag{4.7}$$

If $g = 0$, the operator $\hat{A}$ coincides with the operator $\hat{a}^2$, $\hat{a}$ being the usual oscillator annihilation operator. It is convenient to choose the coordinate eigenfunctions $\langle x|z\rangle$ in the following form (for details see Dodonov *et al*, 1972; 1974a)

$$\langle x|z\rangle = (2\omega_0 x)^{1/2} \exp\left(-\tfrac{1}{2}\omega_0 x^2 + \frac{z}{2}\right) J_b(x[2\omega_0 z]^{1/2});$$

$$\langle z|z^1\rangle = I_b(\sqrt{(z^* z^1)}) \qquad \langle z| \equiv (|z\rangle)^+ \tag{4.8}$$

I_b being the modified Bessel function. One can check using the relations given in the paper by Dodonov *et al.* (1974a) that the operators $\hat{A}$, $\hat{A}^+$ and $\hat{B} = \frac{1}{4}[\hat{A}, \hat{A}^+]$ act on the functions $\psi(z^*)$ according to the following formula

$$\hat{A}\psi(z^*) = 4\left(z^* \frac{d^2}{dz^{*2}} + \frac{d}{dz^*} - \frac{b^2}{4z^*}\right)\psi(z^*);$$

$$\hat{A}^+\psi(z^*) = z^*\psi(z^*);$$

$$\hat{B}\psi(z^*) = \left(1 + 2z^* \frac{d}{dz^*}\right)\psi(z^*);$$

$$[\hat{B}, \hat{A}] = -2\hat{A}; \qquad [\hat{B}, \hat{A}^+] = 2\hat{A}^+ \tag{4.9}$$

Evidently, these operators are generators of the group $SU(1, 1)$ (see also the paper by Barut and Girardello, 1971). It can be shown that the following relation is valid:

$$\psi(z^*) = \int \psi(y^*) I_b(\sqrt{(z^* y)})\, d\mu(y);$$

$$d\mu(y) = \frac{1}{2\pi} K_b(|y|)\, d\,\mathrm{Re}\, y\, d\,\mathrm{Im}\, y \tag{4.10}$$

K_B being the Mcdonald function (the modified Bessel function of the third kind). Therefore the kernel of the unit operator is $G(z^*, y; 0) = I_b(\sqrt{(z^* y)})$. Now we can apply the general scheme of the first section to calculate the Green function in the z-representation. One can check that the operators $\hat{U}\hat{A}\hat{U}^{-1}$, $\hat{U}\hat{A}^+\hat{U}^{-1}$ and $\hat{U}\hat{B}\hat{U}^{-1}$ are linear functions of the operators $\hat{A}, \hat{A}^+$ and $\hat{B}$, as well as the operators $\hat{Q}_j$ are linear functions of the operators $\hat{q}_j$. Conse-

quently equation (2.13) is reduced in the case under study to the Bessel equation, so that the Green function is

$$G(z^*, y; t) = \xi^{-1} I_b(\xi^{-1}\sqrt{(z^* y)}) \exp\left[\frac{1}{2\xi}(\eta z^* - \eta^* y)\right] \tag{4.11}$$

$\xi(t)$ and $\eta(t)$ are the solutions to equations (4.12b) with the coefficients

$$d_0(t) = \frac{\omega_0}{2}(1 + \omega^2(t)/{\omega_0}^2)$$

$$d_1(t) = \frac{\omega_0}{2}(1 - \omega^2(t)/{\omega_0}^2)$$

The multidimensional singular oscillator with the potentials of the type $g(\vartheta, \varphi)/r^2$ was considered by Dodonov *et al.* (1974b).

5. *Green Functions of the Stationary Schrödinger Equations*

In this Section we consider the Hamiltonians which do not depend on time. The Green function of the stationary Schrödinger equation satisfies the equation

$$(\hat{H} - E)G(\mathbf{x}_2, \mathbf{x}_1; E) = \delta(\mathbf{x}_2 - \mathbf{x}_1) \tag{5.1}$$

Since the time-dependent Green function $G(\mathbf{x}_2, \mathbf{x}_1; t)$ satisfies the equation

$$\left(i\hbar\frac{\partial}{\partial t} - \hat{H}\right)G(\mathbf{x}_2, \mathbf{x}_1; t) = i\ \ \delta(t)\delta(\mathbf{x}_2 - \mathbf{x}_1)$$

$$G(t < 0) = 0 \tag{5.2}$$

the function $G(E)$ can be obtained by means of the Fourier transformation of the function $G(t)$

$$G(\mathbf{x}_2, \mathbf{x}_1; E) = \frac{i}{\hbar}\int_0^\infty G(\mathbf{x}_2, \mathbf{x}_1; t)e^{i(Et/\hbar)}\,dt \tag{5.3}$$

(or by means of the Laplace transformation if one makes the substitution $t = -i\hbar\tau$; note that the integral (5.3) may converge not for all values of the parameter E, but this is not essential because $G(E)$ is the analytical function of E and can be obtained with the aid of the analytical continuation). We want to consider several important particular cases when the function $G(E)$ can be calculated exactly. The first example is the N-dimensional isotropic oscillator in the coherent states representation: $\hat{H} = \frac{1}{2}(\hat{a}^+\hat{a} + \hat{a}\hat{a}^+)$; $\hbar = 1$. The time-dependent Green function is (see equation (3.14))

$$G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; t) = \exp\left(\frac{-it}{2} + \boldsymbol{\alpha}^*\boldsymbol{\beta}e^{-it}\right) \tag{5.4}$$

Using the formula 4.5 (36) from the tables by Erde'lyi *et al.* (1954) one can obtain the following result:

$$G(\boldsymbol{\alpha}^*, \boldsymbol{\beta}; E) = (\tfrac{1}{2} - E)^{-1} e^{\boldsymbol{\alpha}^* \boldsymbol{\beta}} \phi(1; \tfrac{3}{2} - E; -\boldsymbol{\alpha}^* \boldsymbol{\beta}) \tag{5.5}$$

$\phi(a; b; z)$ is the confluent hypergeometric function. This formula in the case $N = 1$ was obtained by Dimashko and Granovsky (1974). The function $G(E)$ in the coordinate representation was calculated by Bakhrakh *et al.* (1972).

Let us consider the stationary singular oscillator in the z-representation. The function $G(t)$ is given by equation (4.11) with $\xi(t) = \exp(i\omega t)$ and $\eta(t) = 0$. The function $G(E)$ can be obtained by the Mellin transformation if one makes the substitutions $t = -i\tau$, $\operatorname{Im} \tau = 0$, $\tau > 0$ and $x = e^{-\omega\tau}$, $0 < x < 1$ in equation (5.3).

Then the formula 6.8(2) from (Erde'lyi *et al.*, 1954) leads to the following result:

$$G(z^*, y; E) = [2^b \omega \Gamma(b+1)(b+1-E/\omega)]^{-1} \times$$
$$\times {}_1F_2(\tfrac{1}{2}(b+1-E/w); b+1, \tfrac{1}{2}(b+3-E/w); \tfrac{1}{4}z^* y) \tag{5.6}$$

(for the definition of the function ${}_1F_2$ see Erde'lyi *et al.*, 1954). As the last example we consider the two-dimensional Schrödinger equation with the Hamiltonian

$$H = \frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2, \qquad \mathbf{A} = \tfrac{1}{2}\mathscr{H}(-y, x)$$

describing the motion of a charged particle in the plane xy perpendicular to the uniform magnetic field $\mathscr{H}$. The function $G(t)$ is well known in this case (see e.g. Feynman and Hibbs, 1965), so that we do not write it here. The integral (5.3) can be calculated if one takes into account equation 4.5 (41) from (Erde'lyi *et al*, 1954).

So the two-dimensional stationary Green Function is

$$G(\mathbf{z}_2, \mathbf{z}_1; E) = \frac{m}{2\pi\hbar^2}\Gamma(\tfrac{1}{2} - \tilde{E}) f(\mathbf{z}_2, \mathbf{z}_1) \rho^{-1} W_{\tilde{E}, 0}(\rho^2);$$

$$\rho^2 = \frac{m\omega}{2\hbar}(\mathbf{z}_2 - \mathbf{z}_1)^2; \qquad \mathbf{z} \equiv (x, y); \qquad \tilde{E} = E/\hbar\omega; \tag{4.7}$$

$$\omega = e\mathscr{H}/mc; \qquad f(\mathbf{z}_2, \mathbf{z}_1) = \exp\left[\frac{im\omega}{2\hbar}(x_1 y_2 - x_2 y_1)\right]$$

W is the Whittaker function. The stationary Green function in the three-dimensional case was obtained by Gountaroulis (1972). The poles of the functions $G(E)$ in all cases yield energy levels. For the residues at these points in the last case one can obtain the following expression:

$$res\, G(E_n) = \sum_{E_\alpha = E_n} \varphi_\alpha(z_2)\varphi_\alpha^*(z_1) = \frac{m\omega}{2\pi\hbar} f(z_2, z_1) \exp(-\tfrac{1}{2}\rho^2) L_n(\rho^2) \tag{5.8}$$

L_n is the Laguerre polynomial, and the summation in this equation is performed over all eigenstates with the same energy $E_n = \hbar\omega(n + \frac{1}{2})$, $n = 0, 1, 2, \ldots$

6. *Conclusion*

The method of constructing the Green functions with the aid of integrals of the motion developed in the present article can be successfully applied for solving many interesting physical problems, for example, for constructing the density matrices of both equilibrium and nonequilibrium quantum systems with quadratic Hamiltonians (Dodonov *et al.*, 1974c). In the case of non-quadratic systems this method can be a useful tool for the approximate calculation of the density matrix. This method has many common features with the path integrals method by Feynman, but it seems to us to be more simple from the viewpoint of calculations. It can be generalized to the case of relativistic equations of Dirac's and Klein–Gordon's types, if one uses the proper time method. In particular, this method enables to calculate the Green function of the relativistic particle in external electromagnetic fields obtained earlier by Batalin and Fradkin (1970) with the aid of the path integrals method.

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